a review of A counterexample to the weak density of smooth maps between manifolds in Sobolev spaces by Bethuel, Fabrice

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A counterexample to the weak density of smooth maps between manifolds in Sobolev spaces. (English) Zbl 07160156

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Let M and N be two manifolds with $\mathcal{N}$ isometrically embedded in some euclidean space $\mathbb{R}^l$ and $\mathcal{M}$ possibly having nonempty boundary. For given numbers $0 < s < \infty$ and $1 \leq p < \infty$, this paper is concerned with the Sobolev space

$$W^{s,p}(\mathcal{M},\mathcal{N}) = \{u \in W^{s,p}(\mathcal{M},\mathbb{R}^l) \mid u(x) \in N \text{ for almost every } x \in M\}.$$

The study of these spaces is motivated particularly by various problems in physics, such as liquid crystal theory, Yang-Mills-Higgs or Ginzburg-Landau models, in which singularities of topological nature loom, yielding maps which are not continuous but lie in suitable Sobolev spaces, built up in view of the corresponding variational frameworks. The seminal work of *R. Schoen* and *K. Uhlenbeck* [J. Differ. Geom. 18, 253–268 (1983; Zbl 0547.58020)] has sparked rapid development in research of this field for last decades. The approximation of maps in $W^{s,p}(\mathcal{M},\mathcal{N})$ by smooth maps or maps with singularities of prescribed type lies central. The author restricts himself to the case $s = 1$ in the paper. It turns out that when $1 \leq p < \dim \mathcal{M}$, the answer to the approximation problem is strongly related to the $[p]$-th homotopy group $\pi_{[p]}(\mathcal{N})$ of the target manifold $\mathcal{N}$, where [ ] stands for the Gauss symbol.

This paper is concerned with approximation by sequences of smooth maps at the level of weak convergence.

It was observed in [*F. Bethuel*, Acta Math. 167, No. 3–4, 153–206 (1991; Zbl 0756.46017)] that

Theorem. Assume that

$$1 \leq p < m$$

and

$$\pi_{[p]}(\mathcal{N}) \neq 0$$

Then $C^\infty(\mathbb{B}^m,\mathcal{N})$ is not sequentially weakly dense in $W^{1,p}(\mathbb{B}^m,\mathcal{N})$.

It was shown in [*F. Bethuel*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 7, No. 4, 269–286 (1990; Zbl 0708.58004); Acta Math. 167, No. 3–4, 153–206 (1991; Zbl 0756.46017); *F. Bethuel* and *X. Zheng*, J. Funct. Anal. 80, No. 1, 60–75 (1988; Zbl 0657.46027)] that

Theorem. Let p be an integer. Given any manifold $\mathcal{M}$, $C^\infty(\mathcal{M},\mathbb{S}^p)$ sequentially weakly dense in $W^{1,p}(\mathcal{M},\mathbb{S}^p)$.

A positive answer was given in [*P. Hajlasz*, Nonlinear Anal., Theory Methods Appl. 22, No. 12, 1579–1591 (1994; Zbl 0820.46028)] for $(p-1)$-connected manifolds $\mathcal{N}$ and in [*M. R. Pakzad* and *T. Rivière*, Geom. Funct. Anal. 13, No. 1, 223–257 (2003; Zbl 1028.58008)] in the case $p = 2$ for whatever manifold $\mathcal{N}$. Similar results involving H^2 energy were given in [*R. Hardt* and *T. Rivière*, Calc. Var. Partial Differ. Equ. 54, No. 3, 2713–2749 (2015; Zbl 06505905)].

The main result of the paper is

Theorem. Given any manifold $\mathcal{M}$ of dimension larger or equal to 4, $C^\infty(\mathcal{M},\mathbb{S}^2)$ is not sequentially weakly dense in $W^{1,3}(\mathcal{M},\mathbb{S}^2)$.

The author explicitly constructs a map which is not weakly approximable in $W^{1,3}(\mathcal{M},\mathbb{S}^2)$ by maps in $C^\infty(\mathcal{M},\mathbb{S}^2)$. One of the central ingredients in the argument is strongly related to issues in branched transportation and irrigation theory in the critical exponential case. When the domain $\mathcal{M}$ has a more sophisticated topology, it was shown in [*F. Hang* and *F. Lin*, Math. Res. Lett. 8, No. 3, 321–330 (2001; Zbl 1049.46018); Acta Math. 191, No. 1, 55–107 (2003; Zbl 1061.46032)] that the topology of $\mathcal{M}$ might induce some additional obstructions to the approximation problem.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

58C25 Differentiable maps on manifolds
46E35 Sobolev spaces and other spaces of "smooth" functions, embedding theorems, trace theorems

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